

OUTCOMES OF THE ABEL IDENTITY

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ABSTRACT. Through symbolic methods, we state explicit formulae for Tchebychev, Gegenbauer, Meixner, Mittag-Leffler, and Pidduck polynomials. This is done by underlining the crucial role played by the Abel identity in revisiting the Lagrange inversion formula and the theory of the Riordan arrays.

Keywords: umbral calculus, Abel polynomials, Lagrange inversion formula, Sheffer sequences, Riordan arrays, orthogonal polynomials.

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1. INTRODUCTION

Given a variable x and a complex number a , then the sequence $(a_n(x, a))_{n \geq 0}$ of Abel polynomials is defined by $a_n(x, a) = x(x + na)^{n-1}$. They hold some interesting combinatorial insight [19], and they satisfy nice identities such as

$$(1) \quad (x + y)^n = \sum_{k=0}^n \binom{n}{k} a_k(x, a)(y - ka)^{n-k},$$

$$(2) \quad a_n(x + y) = \sum_{k=0}^n \binom{n}{k} a_k(x, a)a_{n-k}(y, a),$$

$$(3) \quad D_x a_n(x, a) = n a_{n-1}(x + a, a).$$

In particular, in [3] the Abel identity (1) and the binomial identity (2) are obtained in a simple way by means of the key-property (3). The polynomial sequences $(p_n(x))_{n \geq 0}$, with $\deg p_n = n$, satisfying $p_n(x + y) = \sum_{k=0}^n \binom{n}{k} p_k(y)p_{n-k}(y)$ are often called polynomials of binomial type, or binomial sequences. So, because of (2) we see that $(a_n(x, a))_{n \geq 0}$ is of binomial type. Some investigation on the recursive properties of the matrix of the coefficients of a binomial sequence was carried out by Knuth [10]. The wider class of Sheffer polynomials, that includes binomial sequences, have been deeply studied by Roman [14], through the “modern” umbral calculus originated with the seminal paper of Rota [15]. In 1994, Rota and Taylor [18] presented a renewed version of the umbral calculus which is based on the notion of “umbra”. They called it the “classical umbral calculus”. Essentially, this symbolic method consists of an infinite set $\mathcal{A} = \{\alpha, \beta, \gamma, \dots\}$ of variables, called umbrae, and of the linear functional evaluation $E: \mathbb{C}[\mathcal{A}] \rightarrow \mathbb{C}$ holding the uncorrelation property $E[\alpha^n \beta^m \dots \gamma^l] = E[\alpha^n]E[\beta^m] \dots E[\gamma^l]$. Roughly speaking,

we pass from the umbral calculus to the classical umbral calculus by associating each linear functional $L: \mathbb{C}[x] \rightarrow \mathbb{C}$ with the umbra α satisfying $L[x^n] = E[\alpha^n]$. The alphabet \mathcal{A} is extended by adding copies α', α'', \dots of each umbra α , and by introducing new symbols called auxiliary umbrae. More precisely, if k is a positive integer then $k.\alpha$ is the auxiliary umbra defined to satisfy $E[(k.\alpha)^n] = E[(\alpha' + \alpha'' + \dots + \alpha^{(k)})^n]$, for all $n \geq 0$, and the umbral Abel polynomial $\mathfrak{a}_n(\gamma, \alpha) = \gamma(\gamma + n.\alpha)^{n-1}$ may be considered. By setting $\gamma = y$, $\alpha = a$ and by assuming $E[y^n] = y^n$ for all $n \geq 0$, we get $E[\mathfrak{a}_n(y, a)] = a_n(y, a)$, or more compactly $\mathfrak{a}_n(y, a) \simeq a_n(y, a)$. By means of umbral Abel polynomials, Rota, Shen and Taylor [17] have obtained a very nice characterization of the polynomials of binomial type: these polynomial sequences $(p_n(x))_{n \geq 0}$ precisely are the sequences such that $p_n(x) \simeq \mathfrak{a}_n(x, \alpha)$ for all $n \geq 0$, for a suitable α . Hence, from an umbral point of view all polynomial sequences of binomial type have the form of an (umbral) Abel polynomial. In 2001, Taylor [21] showed that Abel polynomials encode not only binomial sequences, but the whole class of Sheffer sequences: $(s_n(x))_{n \geq 0}$ is a Sheffer sequence if and only if $s_n(x) \simeq \mathfrak{a}_n(x + \gamma, \alpha)$. In the same year Di Nardo and Senato [7] began their exploration of the classical umbral calculus, by enriching the syntax and by stating for the first time the umbral Abel identity. Then, with Niederhausen [5] they have also given a more complete description of the Sheffer sequences, by underlining connections with the Lagrange Inversion Formula, and by showing the way to move towards the Riordan arrays theory [2, 4, 20, 22, 23]. A deeper study of the Riordan arrays by means of Rota's umbrae can be found in [1]. In the last ten years, a classical umbral calculus approach to the classical cumulant theory have been carried out [8, 16]. Abel polynomials come back on the scene in a paper by Di Nardo, Petruccio and Senato [6], where they play a crucial role in defining a unifying framework for classical, boolean and free cumulants, and provide a bridge with the combinatorics of parking functions through the volume polynomial of Pitman and Stanley [13]. See also [12] on this subject. Finally, a wider range of identities encoded by the umbral Abel polynomials can be found in [11].

In this paper, we present an elementary proof of the umbral Abel identity, by showing that (3) generalizes to umbral Abel polynomials. From this starting point, we show how some fundamental facts relating Sheffer sequences, Lagrange Inversion Formula and Riordan arrays arise in a trivial way and with one-line proofs. Moreover, we give an application of this methods in determining explicit formulae for the following class of polynomials: Tchebychev (II kind), Gegenbauer, Meixner (I kind), Mittag-Leffler and Pidduck.

2. THE ABEL IDENTITY AND THE LAGRANGE INVERSION FORMULA.

Let $\mathbf{x} \cup \{z\}$ be a finite set of commuting variables. Denote by $\mathbb{C}[\mathbf{x}]$ the ring of polynomials with complex coefficients in the variables of \mathbf{x} , and by $\mathbb{C}[\mathbf{x}][[z]]$ the ring of all formal power series (f.p.s) of type

$$f(z) = \sum_{n \geq 0} p_n(\mathbf{x}) \frac{z^n}{n!},$$

with $p_n(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ for all $n \geq 0$. As is customary, the coefficient of z^n in a f.p.s. $f(z)$ is denoted by $[z^n]f(z)$, with $[z^0]f(z) = f(0)$. For all $f(z) \in \mathbb{C}[\mathbf{x}][[z]]$ such that $f(0) = 1$, we introduce a symbol α , called the *umbra* of $f(z)$. The set \mathcal{A} of all umbrae will be called the *base alphabet*. The umbra of 1 is called the *augmentation*

and it is denoted by ε , the umbra of $1+z$ is named the *singleton* and it is denoted by χ , the umbra β of e^{e^z-1} is called the *Bell umbra*. Then, the *evaluation* is the only $\mathbb{C}[\mathbf{x}]$ -linear functional $E: \mathbb{C}[\mathbf{x}][\mathcal{A}] \rightarrow \mathbb{C}[\mathbf{x}]$ such that

- (1) $E[\alpha^n] = n![z^n]f(z)$, if and only if α is the umbra of $f(z)$,
- (2) $E[\alpha^n \gamma^m \dots \delta^l] = E[\alpha^n]E[\gamma^m] \dots E[\delta^l]$, whenever $\alpha, \gamma, \dots, \delta$ are pairwise distinct umbrae (*uncorrelation property*).

Here, $\mathbb{C}[\mathbf{x}][\mathcal{A}]$ denotes the ring of polynomials in the variables of \mathcal{A} and with coefficients in $\mathbb{C}[\mathbf{x}]$. To avoid any confusion, complex numbers in \mathbb{C} will be denoted by a, b, \dots , polynomials in $\mathbb{C}[\mathbf{x}]$ by $p(\mathbf{x}), q(\mathbf{x}), \dots$, umbrae in \mathcal{A} by Greek letters α, β, \dots , and elements in $\mathbb{C}[\mathbf{x}][\mathcal{A}]$ by $\mathbf{p}, \mathbf{q}, \dots$. Each $\mathbf{p} \in \mathbb{C}[\mathbf{x}][\mathcal{A}]$ will be named an *umbral polynomial*. We say that $p_n(\mathbf{x})$ is the *nth moment* of \mathbf{p} if $E[\mathbf{p}^n] = p_n(\mathbf{x})$, and that the sequence $(p_n(\mathbf{x}))_{n \geq 0}$ is *represented* by \mathbf{p} if $E[\mathbf{p}^n] = p_n(\mathbf{x})$ for all $n \geq 0$. Note that, by construction each sequence such that $p_0(\mathbf{x}) = 1$ is represented by a unique umbra $\alpha \in \mathcal{A}$. Now, we extend the action of E to the f.p.s. of type $e^{\mathbf{p}z}$, so that

$$E[e^{\mathbf{p}z}] = E \left[\sum_{n \geq 0} \mathbf{p}^n \frac{z^n}{n!} \right] = 1 + \sum_{n \geq 1} E[\mathbf{p}^n] \frac{z^n}{n!}.$$

We set $f_{\mathbf{p}}(z) = E[e^{\mathbf{p}z}]$ and say that $f_{\mathbf{p}}(z)$ is *generating function* of \mathbf{p} . Note that $e^{(\alpha+\gamma)z} = e^{\alpha z} e^{\gamma z}$ and because of the uncorrelation property we have $f_{\alpha+\gamma}(z) = f_{\alpha}(z)f_{\gamma}(z)$. Nevertheless, if $n \in \mathbb{Z}$ then $e^{n\alpha z} = (e^{\alpha z})^n$ but $f_{n\alpha}(z) \neq f_{\alpha}(z)^n$. Analogously, $e^{\gamma \alpha z} = e^{\gamma(\log e^{\alpha z})}$ but $f_{\gamma \alpha}(z) \neq f_{\gamma}(\log f_{\alpha}(z))$. Then, in order to preserve some notational coherence, we define new symbols, denoted by $n.\alpha$ and $\gamma.\alpha$, called the *auxiliary umbrae*. A new alphabet \mathcal{B} is defined to contain them, and the action of E is then extended to $\mathbb{C}[\mathbf{x}][\mathcal{A} \cup \mathcal{B}]$. In detail, for all \mathbf{p} and \mathbf{q} the auxiliary umbra $\mathbf{p}.\mathbf{q}$ is defined to satisfy $E[e^{(\mathbf{p}.\mathbf{q})z}] = f_{\mathbf{p}}(\log f_{\mathbf{q}}(z))$, so that $E[e^{(a.\alpha)z}] = f_{\alpha}(z)^a$ and $E[e^{(\alpha.a)z}] = f_{\alpha}(az)$. Iterations of type $\delta.\gamma.\alpha$ are allowed, and if β is the Bell umbra then we obtain $f_{\gamma.\beta.\alpha}(z) = f_{\gamma}(\log f_{\beta}(\log f_{\alpha}(z))) = f_{\gamma}(f_{\alpha}(z) - 1)$. Sometimes, $\gamma.\beta.\alpha$ is named the *composition umbra* of γ with α . Further auxiliary umbrae are denoted by α_D , with $\alpha \in \mathcal{A}$. They are named *derivative umbrae* and are defined by $E[\alpha_D^n] = E[D_{\alpha}^n \alpha^n]$, for all $n \geq 1$, where $D_{\alpha}: \alpha^n \mapsto n\alpha^{n-1}$. Hence, α_D satisfy $f_{\alpha_D}(z) - 1 = z f_{\alpha}(z)$. From now on, we will write $\mathbf{p} \simeq \mathbf{q}$ (\mathbf{p} is *equivalent* to \mathbf{q}) whenever $E[\mathbf{p}] = E[\mathbf{q}]$, and $\mathbf{p} \equiv \mathbf{q}$ (\mathbf{p} is *similar* to \mathbf{q}) if and only if $\mathbf{p}^n \simeq \mathbf{q}^n$ for all $n \geq 0$. Also, we will write $e^{\mathbf{p}z} \simeq f(z)$ instead of $E[e^{\mathbf{p}z}] = f(z)$, and $e^{\mathbf{p}z} \simeq e^{\mathbf{q}z}$ if $E[e^{\mathbf{p}z}] = E[e^{\mathbf{q}z}]$. This provides compact expressions like $\alpha.a \equiv a\alpha$ and $b.a \equiv ba$. In particular, we notice that the restriction of the dot-operation on \mathbb{C} (and $\mathbb{C}[\mathbf{x}]$) can be thought as the usual multiplication on \mathbb{C} . Also, we get the duality relation $\chi.\beta \equiv \beta.\chi \equiv 1$. As it is shown by equivalence (4), the moments of dot-operations of type $\gamma.\beta.\alpha_D$ holds a binomial-like expansion in terms of the moments of α and γ . In fact, since $e^{(\gamma.\beta.\alpha_D)z} \simeq e^{\gamma(z f_{\alpha}(z))}$ then we obtain

$$e^{(\gamma.\beta.\alpha_D)z} \simeq \sum_{i \geq 0} \gamma^i \frac{z^i}{i!} e^{(i.\alpha)z} \simeq \sum_{i \geq 0} \sum_{j \geq 0} \gamma^i (i.\alpha)^j \frac{z^i}{i!} \frac{z^j}{j!}.$$

Finally, we gain

$$(4) \quad (\gamma.\beta.\alpha_D)^n \simeq \sum_{k=0}^n \binom{n}{k} \gamma^k (k.\alpha)^{n-k}.$$

We stress that, for any fixed α there are pairwise distinct auxiliary umbrae of type $1.\alpha, 1.1.\alpha, \dots$, and of type $\alpha.1, \alpha.1.1, \dots$, representing the same sequence of moments (i.e. similar auxiliary umbrae). We will refer to any of them by means of more compact notations such as α', α'', \dots . It follows that $n.\alpha \equiv \alpha' + \alpha'' + \dots + \alpha^{(n)}$, and then $n.\alpha$ behaves as a sum of n distinct similar umbrae. Moreover, to our aim it is not important to distinguish umbrae in \mathcal{A} from the auxiliary umbrae in \mathcal{B} . So that, the term “umbra” will refer to any unspecified symbol in $\mathcal{A} \cup \mathcal{B}$. The pair $(\mathbb{C}[\mathbf{x}][\mathcal{A} \cup \mathcal{B}], E)$ provides a so-called *saturated umbral calculus* over $\mathbb{C}[\mathbf{x}]$. See [7, 18] for more details. From now on, all sums of type $\alpha + (-k.\gamma)$ will be abbreviated by $\alpha - k.\gamma$. Hence, we have $k.\alpha - k.\alpha \equiv -k.\alpha + k.\alpha \equiv \varepsilon \equiv 0$, for all k .

Definition 1. Let α, γ be umbrae, and let n be a positive integer. Then, the umbral polynomial

$$(5) \quad \mathfrak{a}_n(\gamma, \alpha) = \gamma(\gamma + n.\alpha)^{n-1},$$

is called an *umbral Abel polynomial*, or simply an *Abel polynomial*.

If α and γ are replaced by a complex number a and a variable $x \in \mathbf{x}$ respectively, then we gain $\mathfrak{a}_n(x, a) \simeq x(x + na)^{n-1}$ (being $n.a \equiv na$). Hence, the definition above returns the classical Abel polynomials $a_n(x, a) = x(x + na)^{n-1}$. The well-known property $D_x a_n(x, a) = na_n(x + a, a)$ generalizes to $\mathfrak{a}_n(\gamma, \alpha)$ as follows.

Lemma 1. For all umbrae α, γ , and for all $n \geq 1$ we have

$$(6) \quad D_\gamma \mathfrak{a}_n(\gamma, \alpha) \simeq n \mathfrak{a}_{n-1}(\gamma + \alpha', \alpha).$$

Proof. Elementary computations show that

$$(7) \quad D_\gamma \gamma(\gamma + n.\alpha)^{n-1} = n\gamma(\gamma + n.\alpha)^{n-2} + (n.\alpha)(\gamma + n.\alpha)^{n-2}.$$

It is also clear that

$$(8) \quad n\gamma(\gamma + n.\alpha)^{n-2} \simeq n\gamma(\gamma + \alpha + (n-1).\alpha)^{n-2}.$$

Moreover, if $\alpha', \alpha'', \dots, \alpha^{(n)}$ are distinct umbrae similar to α , then we have

$$(9) \quad \begin{aligned} (n.\alpha)(\gamma + n.\alpha)^{n-2} &\simeq (\alpha' + \alpha'' + \dots + \alpha^{(n)})(\gamma + \alpha' + \alpha'' + \dots + \alpha^{(n)})^{n-2} \\ &\simeq \sum_{i=1}^n \alpha^{(i)}(\gamma + \alpha^{(i)} + (n-1).\alpha)^{n-2} \\ &\simeq n\alpha(\gamma + \alpha + (n-1).\alpha)^{n-2}. \end{aligned}$$

The equivalence (6) follows by comparing (7), (8) and (9). \square

Theorem 2 (Umbral Abel identity). For all umbrae α, γ, δ we have

$$(10) \quad (\delta + \gamma)^n \simeq \sum_{k=0}^n \binom{n}{k} (\delta + k.\alpha)^{n-k} \gamma(\gamma - k.\alpha)^{k-1}.$$

Proof. By thinking of $((\delta + \gamma)^n)_{n \geq 0}$ and $(\mathfrak{a}_n(\gamma, -1.\alpha))_{n \geq 0}$ as basis of a suitable ring of umbral polynomials in the distinguished variable γ , we may refer to the unique sequence $(b_n)_{n \geq 0}$ of coefficients such that

$$(11) \quad (\delta + \gamma)^n = \sum_{k=0}^n b_k \mathfrak{a}_k(\gamma, \alpha).$$

Let us apply D_γ^i to both sides in (10). Via (6) we gain

$$(n)_i(\delta + \gamma)^{n-i} \simeq \sum_{k=0}^n b_k(k)_i \mathbf{a}_{k-i}(\gamma - i.\alpha', -1.\alpha),$$

with $(n)_i = n(n-1)\cdots(n-i+1)$. Finally, because of $-i.\alpha + i.\alpha \equiv 0$, by evaluating on $\gamma = k.\alpha'$ both sides in (11) we have $\mathbf{a}_{i-k}(0, \alpha) \simeq \delta_{i,k}$, and then $(n)_k(\delta + k.\alpha)^{n-k} \simeq k! b_k$. \square

As a by product we obtain the following generalization.

Corollary 3. *For all α, γ, δ , and for all polynomials $q(\gamma)$ of degree n we have*

$$(12) \quad q(\delta + \gamma) \simeq \sum_{k=0}^n \frac{D_\gamma^k q(\delta + k.\alpha)}{k!} \gamma(\gamma - k.\alpha)^{k-1},$$

with $D_\gamma^k q(\delta + k.\alpha)$ denoting the k th derivative of $q(\gamma)$ evaluated on $\gamma = \delta + k.\alpha$.

Proof. It follows from (10) because of the linearity of E and of the D_γ^k 's. \square

If $E[\alpha] \neq 0$ then $f_\alpha(z) - 1$ admits compositional inverse $(f_\alpha(z) - 1)^{(-1)}$. Then we denote by $\alpha^{(-1)}$ an umbra such that $f_{\alpha^{(-1)}}(z) - 1 = (f_\alpha(z) - 1)^{(-1)}$, or equivalently such that $\alpha.\beta.\alpha^{(-1)} \equiv \alpha^{(-1)}.\beta.\alpha \equiv \chi$, where χ is the singleton. For all umbrae γ and α , we define $\mathfrak{K}_{\gamma,\alpha}$ to be an auxiliary umbra with moments

$$(13) \quad \mathfrak{K}_{\gamma,\alpha}^n \simeq \gamma(\gamma - n.\alpha)^{n-1}, \text{ for all } n \geq 1.$$

Moreover, we will abbreviate $\mathfrak{K}_{\alpha,\alpha}$ by \mathfrak{K}_α . The umbra \mathfrak{K}_α plays a fundamental role within umbral cumulant theory [6]. Note that, the equivalence (6) may be expressed as

$$(14) \quad \mathfrak{K}_{\gamma_D,\alpha} \equiv (\mathfrak{K}_{\gamma-1,\alpha,\alpha})_D.$$

Theorem 4 (Lagrange Inversion Formula). *For all umbrae α and γ we have*

$$(15) \quad (\gamma.\beta.\alpha_D^{(-1)})^n \simeq \gamma(\gamma - n.\alpha)^{n-1}.$$

Proof. By means of (4), (13), and (10) with $\delta = 0$, we have $\mathfrak{K}_{\gamma,\alpha}.\beta.\alpha_D \equiv \gamma$, and finally $\mathfrak{K}_{\gamma,\alpha} \equiv \gamma.\beta.\alpha_D^{(-1)}$. \square

In terms of generating functions, the equivalence (15) is nothing but one of the most general versions of the famous Lagrange Inversion Formula,

$$[z^n]f_\gamma((zf_\alpha(z))^{(-1)}) = \frac{1}{n}[z^{n-1}]f'_\gamma(z) \left(\frac{1}{f_\alpha(z)} \right)^n.$$

We stress that, in this setting, it is merely a corollary of the Abel identity stated by a one-line proof. Finally, since $\chi \equiv \varepsilon_D$, via (14) and (15) we obtain $\alpha_D^{(-1)} \equiv \mathfrak{K}_{\varepsilon_D,\alpha} \equiv (\mathfrak{K}_{-1,\alpha,\alpha})_D$. However, it is $\mathfrak{K}_{-1,\alpha,\alpha} \equiv -1.\mathfrak{K}_\alpha$, and then we obtain

$$(16) \quad \alpha_D \equiv (-1.\mathfrak{K}_\alpha)_D^{(-1)}.$$

3. EXPLICIT FORMULAE FOR POLYNOMIAL SEQUENCES

Let $x \in \mathbf{x}$. A polynomial sequence $(s_n(x))_{n \geq 0}$ is said to be a *Sheffer sequence* if and only if there are $A(z), B(z) \in \mathbb{C}[[z]]$ with $A(0) \neq 0 \neq B(0)$, and such that

$$(17) \quad \sum_{n \geq 0} s_n(x) \frac{z^n}{n!} = A(z) e^{xzB(z)}.$$

When $A(z) = 1$ the polynomial sequence is said to be a *binomial sequence*, or an *associated sequence*. Sheffer sequences can be also characterized as the sequences $(s_n(x))_{n \geq 0}$, with $\deg s_n(x) = n$, satisfying the following *Sheffer identity*,

$$(18) \quad s_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) s_{n-k}(y),$$

with $(p_n(x))_{n \geq 0}$ being a binomial sequence. Analogously, the binomial sequences are the unique sequences $(p_n(x))_{n \geq 0}$, with $\deg p_n(x) = n$, satisfying the *binomial identity*,

$$(19) \quad p_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(y).$$

Abel polynomials $(a_n(x, a))_{n \geq 0}$ are binomial with $B(z) = (ze^{-az})^{(-1)}$. By iterating (6) we get $D_\gamma^k \mathbf{a}_n(\gamma, \alpha) \simeq (n)_k \mathbf{a}_{n-k}(\gamma + k.\alpha)$. Then, by replacing α with $-1.\alpha$ in (12), and choosing $p(\gamma) = \mathbf{a}_n(\gamma, \alpha)$, we recover the binomial identity of umbral Abel polynomials,

$$\mathbf{a}_n(\gamma + \delta, \alpha) \simeq \sum_{k=0}^n \binom{n}{k} \mathbf{a}_k(\gamma, \alpha) \mathbf{a}_n(\delta, \alpha').$$

In the following, by writing a “Sheffer sequence” or a “binomial sequence” we will always mean a “Sheffer sequence with $A(0) = B(0) = 1$ ” or a “binomial sequence with $B(0) = 1$ ”. It is not difficult to see that we are restricting our analysis to the case $s_n(x) = x^n + \text{lower terms}$. Let γ and α be umbrae such that $e^{\gamma z} \simeq A(z)$ and $e^{\alpha z} \simeq B(z)$, then the umbral translation of (17) is

$$(20) \quad s_n(x) \simeq (\gamma + x.\beta.\alpha_D)^n,$$

and Sheffer sequences are moments of umbrae of type $\gamma + x.\beta.\alpha_D$. Thanks to (4), straightforward computations give

$$(21) \quad s_n(x) \simeq \sum_{k=0}^n \binom{n}{k} (\gamma + k.\alpha)^{n-k} x^k,$$

and the coefficient $s_{n,k}$ of x^k in $s_n(x)$ have the following closed expression,

$$(22) \quad s_{n,k} \simeq \binom{n}{k} (\gamma + k.\alpha)^{n-k}.$$

From now on, the polynomial sequence $(s_n(x))_{n \geq 0}$ defined by (20) will be named the *Sheffer sequence* of (γ, α) . Moreover, we will write

$$(\gamma, \alpha) = (s_n(x))_{n \geq 0}$$

to express that one of the equivalent identities (20), (21), and (22) hold. The *Appell sequences* are the Sheffer sequences of (γ, ε) , the *binomial sequences*, also named *associated sequences* [14], are the Sheffer sequences of (ε, α) . If we lift to an

umbral level, then we may appreciate that Sheffer sequences have the form of Abel polynomials.

Theorem 5 (Abel representation of Sheffer sequences). *A polynomial sequence $(s_n(x))_{n \geq 0}$ is the Sheffer sequence of (γ, α) if and only if*

$$s_n(x) \simeq (x + \mathfrak{K}_{\gamma, \alpha}) (x + \mathfrak{K}_{\gamma, \alpha} + n \cdot \mathfrak{K}_{\alpha})^{n-1}.$$

Proof. We have $\gamma + x \cdot \beta \cdot \alpha_D \equiv \mathfrak{K}_{\gamma, \alpha} \cdot \beta \cdot \alpha_D + x \cdot \beta \cdot \alpha_D \equiv (x + \mathfrak{K}_{\gamma, \alpha}) \cdot \beta \cdot (-1 \cdot \mathfrak{K}_{\alpha})_D^{(-1)}$, and the claim follows via (15). \square

Corollary 6 (Appell sequences). *A sequence $(s_n(x))_{n \geq 0}$ is the Appell sequence of (γ, ε) if and only if*

$$s_n(x) \simeq (x + \gamma)^n.$$

Proof. Set $\alpha = \varepsilon$ in the theorem above. \square

Corollary 7 (Binomial sequences). *A sequence $(s_n(x))_{n \geq 0}$ is the binomial sequence of (ε, α) if and only if*

$$s_n(x) \simeq x(x + n \cdot \mathfrak{K}_{\alpha})^{n-1}.$$

Proof. Set $\gamma = \varepsilon$ in the theorem above. \square

Given the Sheffer sequences $(\gamma, \alpha) = (s_n(x))_{n \geq 0}$ and $(\eta, \delta) = (r_n(x))_{n \geq 0}$, one may consider the so-called “umbral composition”

$$(\gamma, \alpha)(\eta, \delta) = (s_n(\mathbf{r}(x)))_{n \geq 0},$$

which is the polynomial sequence defined by [14]

$$s_n(\mathbf{r}(x)) = \sum_{k=0}^n s_{n,k} r_k(x).$$

Being $s_n(x) \simeq (\gamma + x \cdot \beta \cdot \alpha_D)^n$ and $r_n(x) \simeq (\eta + x \cdot \beta \cdot \delta_D)^n$, then the umbra representing $(s_n(\mathbf{r}(x)))_{n \geq 0}$ is obtained by replacing x with $\eta + x \cdot \beta \cdot \delta_D$ in $\gamma + x \cdot \beta \cdot \alpha_D$. Hence, we get

$$s_n(\mathbf{r}(x)) \simeq (\gamma + \eta \cdot \beta \cdot \alpha_D + x \cdot \beta \cdot \delta_D \cdot \beta \cdot \alpha_D)^n.$$

Via generating functions it is quite simple to see that $\delta_D \cdot \beta \cdot \alpha_D \equiv (\alpha + \delta \cdot \beta \cdot \alpha_D)_D$, and the following theorem is proved.

Theorem 8 (Umbral composition of Sheffer sequences). *The umbral composition of the Sheffer sequence of (γ, α) with the Sheffer sequence of (η, δ) is the Sheffer sequence of $(\gamma + \delta \cdot \beta \cdot \alpha_D, \alpha + \delta \cdot \beta \cdot \alpha_D)$. In symbols, we have*

$$(23) \quad (\gamma, \alpha)(\eta, \delta) = (\gamma + \eta \cdot \beta \cdot \alpha_D, \alpha + \delta \cdot \beta \cdot \alpha_D).$$

Note that, $(\varepsilon, \varepsilon) = (x^n)_{n \geq 0}$ is the identity with respect to the umbral composition. Let $A(z), B(z) \in \mathbb{C}[[z]]$ with $A(0) \neq 0 \neq B(0)$. The *exponential Riordan array* defined by $(A(z), B(z))$ is the infinite lower triangular matrix $(s_{n,k})_{n,k}$ determined by

$$(24) \quad s_{n,k} = n! [z^n] A(z) \frac{(zB(z))^k}{k!}.$$

As we have done for Sheffer sequences, we restrict ourselves to the case $A(0) = B(0) = 1$, so that there are α and γ such that $f_\gamma(z) = A(z)$ and $f_\alpha(z) = B(z)$. This way, we obtain

$$(25) \quad s_{n,k} \simeq \binom{n}{k} (\gamma + k.\alpha)^{n-k}.$$

The Riordan array (25) will be named the *Riordan array* of (γ, α) . The following relation among Sheffer sequences and Riordan arrays comes trivially.

Theorem 9. *The infinite lower triangular matrix $(s_{n,k})_{n,k}$ is the Riordan array of (γ, α) if and only if the polynomial sequence $(s_n(x))_{n \geq 0}$ defined by*

$$s_n(x) = \sum_{k=0}^n s_{n,k} x^k$$

is the Sheffer sequence of (γ, α) .

Proof. It follows by comparing (22) and (25). \square

The Fundamental Theorem of the Riordan arrays [24] can be easily stated by replacing x with a umbra η in (18) and (21).

Theorem 10 (Fundamental Theorem of the Riordan arrays). *Let $(s_{n,k})_{n,k}$ be the Riordan array of (γ, α) and let $(a_n)_{n \geq 0}$ be the sequence represented by η . Then we have*

$$(s_{n,k})_{n,k} \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}$$

if and only if $(b_n)_{n \geq 0}$ is represented by $\gamma + \eta.\beta.\alpha_D$.

Let

$$s_n(x) = \sum_{k=0}^n s_{n,k} x^k \text{ and } r_n(x) = \sum_{k=0}^n r_{n,k} x^k.$$

It is plain that

$$s_n(r(x)) = \sum_{k=0}^n \left(\sum_{i=k}^n s_{n,i} r_{i,k} \right) x^k.$$

However, $\sum s_{n,i} r_{i,k}$ is nothing but the (n, k) -entry in the array $(s_{n,k})_{n,k} (r_{n,k})_{n,k}$. This way, the identity (23) also encodes the multiplication of Riordan arrays: the product of the Riordan array of (γ, α) with the Riordan array of (η, δ) is the Riordan array of $(\gamma + \delta.\beta.\alpha_D, \alpha + \delta.\beta.\alpha_D)$. Clearly, the identity is given by $(\varepsilon, \varepsilon)$. Moreover, being $s_{n,n} = 1$ for all $n \geq 0$, we see that $(s_{n,k})_{n,k}$ is invertible and the set of all the Riordan arrays of type (25) is a group with respect to the usual matrix multiplication. Thanks to (23), we may write

$$(\gamma, \alpha)^{-1} = (-1.\mathfrak{K}_{\gamma,\alpha}, -1.\mathfrak{K}_\alpha),$$

where $(\gamma, \alpha)^{-1}$ denotes the unique array such that

$$(\gamma, \alpha)^{-1}(\gamma, \alpha) = (\gamma, \alpha)(\gamma, \alpha)^{-1} = (\varepsilon, \varepsilon).$$

This means that the Riordan array of $(-1.\mathfrak{K}_{\gamma,\alpha}, -1.\mathfrak{K}_\alpha)$ is the inverse of the Riordan array of (γ, α) , or equivalently, that the umbral composition of the Sheffer sequence of (γ, α) with the Sheffer sequence of $(-1.\mathfrak{K}_{\gamma,\alpha}, -1.\mathfrak{K}_\alpha)$ is $(x^n)_{n \geq 0}$.

Remark 1. Classically, the (ordinary) Riordan array $(a_{n,k})_{n,k}$ of $(f(z), g(z))$ is defined by

$$a_{n,k} = [z^n] f(z) (zg(z))^k,$$

with $f(0) \neq 0 \neq g(0)$. Without loss of generality, let $f(0) = 1 = g(0)$ and assume $e^{\gamma z} \simeq f(z)$ and $e^{\alpha z} \simeq g(z)$. We have

$$a_{n,k} \simeq \frac{(\gamma + k\alpha)^{n-k}}{(n-k)!}.$$

If $(s_{n,k})_{n,k}$ is defined by (25) then it is plain that the map $(a_{n,k})_{n,k} \mapsto (s_{n,k})_{n,k}$ acts linearly with respect to the matrix multiplication. Indeed, we have

$$\binom{n}{k} (\sigma + k\rho)^{n-k} \simeq \sum_{i=0}^n \binom{n}{i} (\gamma + i\alpha)^{n-i} \binom{i}{k} (\eta + k\delta)^{i-k}$$

if and only if

$$\frac{(\sigma + k\rho)^{n-k}}{(n-k)!} \simeq \sum_{i=0}^n \frac{(\gamma + i\alpha)^{n-i}}{(n-i)!} \frac{(\eta + k\delta)^{i-k}}{(i-k)!}.$$

Finally, since the exponential Riordan group is isomorphic to the group of Sheffer sequences (with umbral composition as operation), then such an isomorphism extends immediately to ordinary Riordan arrays. This way we recover one of the main results of [9].

Now, let x, y be commuting variables, and let q, t be two parameters. Set $\bar{u} = -1 \cdot \chi - 1$ and consider the following umbra,

$$(26) \quad t \cdot \bar{u} + y \cdot \chi \cdot x \cdot \beta \cdot (q \cdot \bar{u})_D.$$

If $(P_n(x, y; q, t))_{n \geq 0}$ is its sequence of moments, then from (20) and (21), we obtain

$$P_n(x, y; q, t) \simeq \sum_{k=0}^n \binom{n}{k} (t \cdot \bar{u} + kq \cdot \bar{u})^{n-k} (y \cdot \chi \cdot x)^k.$$

Moreover, since

$$\frac{(t \cdot \bar{u} + kq \cdot \bar{u})^{n-k}}{(n-k)!} \simeq \frac{((t + kq) \cdot \bar{u})^{n-k}}{(n-k)!} \simeq \binom{n-k+t+kq-1}{n-k},$$

and being

$$\frac{(y \cdot \chi \cdot x)^k}{k!} \simeq \binom{y}{k} x^k,$$

then the following explicit formula for $P_n(x, y; q, t)$ can be derived,

$$(27) \quad P_n(x, y; q, t) = n! \sum_{k=0}^n \binom{n-k+t+kq-1}{n-k} \binom{y}{k} x^k.$$

Thanks to (26), the generating function of $(P_n(x, y; q, t))_{n \geq 0}$ is

$$1 + \sum_{n \geq 1} P_n(x, y; q, t) \frac{z^n}{n!} = \frac{1}{(1-z)^t} \left(1 + \frac{xz}{(1-z)^q} \right)^y.$$

The polynomial sequence $(P_n(x, y; q, t))_{n \geq 0}$ specializes in well-known families of polynomials, possibly not Sheffer sequences, when the variables x, y , and the parameters q, t are chosen in a suitable way. This way, from (27) we get easily their expansion in terms of some classical basis.

Example 1 (Tchebychev polynomials of the II kind). These are orthogonal polynomials $(U_n(x))_{n \geq 0}$ with generating function

$$\sum_{n \geq 1} U_n(x) z^n = \frac{1}{1 - 2xz + z^2}.$$

They are obtained by setting $(x, y; q, t) = (-2x + 2, -1; 2, 2)$. identity (27) gives the expansion of $T_n(x)$ in terms of the basis $((x - 1)^n)_{n \geq 0}$,

$$U_n(x) = \sum_{k=0}^n \binom{n+k+1}{n-k} 2^k (x-1)^k.$$

Example 2 (Gegenbauer polynomials). These are orthogonal polynomials $(P_n^{(\lambda)}(x))_{n \geq 0}$ with generating function

$$\sum_{n \geq 1} P_n^{(\lambda)}(t) x^n = \left(\frac{1}{1 - 2xz + z^2} \right)^\lambda.$$

Of course, they generalize the Tchebychev polynomials of the II kind and are obtained by setting $(x, y; q, t) = (-2x - 2, -\lambda; 2, 2\lambda)$. identity (27) gives

$$P_n^{(\lambda)}(x) = \sum_{k=0}^n \binom{n+2\lambda+k-1}{n-k} \binom{\lambda+k-1}{k} 2^k (x-1)^k.$$

Example 3 (Meixner of I kind). This orthogonal Sheffer sequence has generating function

$$\sum_{n \geq 0} \mathbf{m}_n(x; b, c) \frac{z^n}{n!} = \left(\frac{1}{1-z} \right)^b \left(\frac{1 - \frac{1}{c}z}{1-z} \right)^x,$$

with $c \neq 0, 1$ and $b \neq 0, -1, -2, \dots$. So, we recover them when $(x, y; q, t) = ((c-1)/c, x; 1, b)$ and obtain

$$\mathbf{m}_n(x; b, c) = n! \sum_{k=0}^n \binom{n+b-1}{n-k} \left(\frac{1-c}{c} \right)^k \binom{x}{k}.$$

Example 4 (Mittlag-Leffler and Pidduck polynomials). The Mittlag-Leffler polynomials $(M_n(x))_{n \geq 0}$ are the following binomial sequence

$$\sum_{n \geq 0} M_n(x) \frac{z^n}{n!} = \left(\frac{1+z}{1-z} \right)^x.$$

Hence, they can be thought as $M_n(x) = \mathbf{m}_n(x; 0, -1)$ and we get them by setting $(x, y; q, t) = (2, x; 1, 0)$. Their explicit formula in terms of the binomial basis $\left(\binom{x}{n} \right)_{n \geq 0}$ is

$$M_n(x) = n! \sum_{k=1}^n \binom{n-1}{n-k} 2^k \binom{x}{k}.$$

Finally, Pidduck polynomials arise as the Sheffer sequence given by

$$\sum_{n \geq 0} P_n(x) \frac{z^n}{n!} = \frac{1}{1-z} \left(\frac{1+z}{1-z} \right)^x.$$

We set $(x, y; q, t) = (2, x; 1, 1)$ and obtain

$$P_n(x) = n! \sum_{k=0}^n \binom{n}{n-k} 2^k \binom{x}{k}.$$

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